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Explorability of a Turbulent Scalar Field

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Abstract— **In this paper we extend the concept of explorability of noisy scalar fields to turbulent scalar fields. Since turbulent fields are prone to sporadic measurement, we use expected value of rate of measurements to estimate the rate of measurement at a location and use this estimate to define explorability of a turbulent scalar field. We define the notion of coherent steps in a turbulent field and prove that if the turbulent field is locally explorable the step taken by the robot will be coherent. We also present a method to compute the Cumulative Distribution Function of the error in estimation of rate of measurement and then show that the probability of a coherent step is inversely proportional to the variance in estimation error if the error has Gaussian distribution.**

I. Introduction

Mobile agents moving in space often encounter scalar or vector fields in their path. A scalar or vector field such as chemical plume, temperature, light intensity etc is generated by a source such as a chemical substance, a fire, or a light source respectively. The field generated by a source can be smooth or turbulent. The smooth fields are characterized by smooth local gradients whereas gradient of a turbulent field is not well defined. In this paper we study the turbulent fields because most of the practical problems or applications in engineering and biology encounter turbulent fields [1]. For example, an underwater vehicle in shallow depths will encounter turbulent flow of water, a robot near an active volcano will encounter turbulent heat waves and a moth will encounter a turbulent chemical plume set up by a mating partner.

A robot carrying sensors for measurement (or detection) can move in an area to measure the value of the field at different locations. The trajectory of the robot is designed depending upon the need of the user or researcher. For example, if a robot has been deployed for surveillance then the trajectory of the robot is designed for the exploration of the field [2]–[5], if a robot is deployed for target search then the trajectory of the robot is designed for localizing the source of the measurement using gradient ascent/descent techniques [6]–[11] in case of smooth fields or gradient-free methods [12]–[15] for the case of turbulent fields. Most of the research work in this field is focused on designing strategies to either explore the field efficiently or search the source under some constraints. However, in this paper we aim to investigate how difficult is it to explore a turbulent scalar field.

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For any application, we would like the measurement to be consistent with the true field. In other words, if the true field increases from location A to location B then the value of measurement at B should be more than the value of measurement at A and vice versa. This need of the consistency of measurement with the true field leads us to the notion of explorability of a field. In [16] the concept of explorability of a noisy scalar field was introduced. It was shown that the probability of a false walk (inconsistent measurement w.r.t. true field) is less than 0.5 if the field is locally explorable. In [17] the concept of explorability was extended to Gaussian scalar fields and source seeking behavior of the robots was analyzed by the definition of coherent and incoherent steps. Since, the turbulent fields are prone to sporadic measurements, the concept of explorability can not be borrowed in its original form [16], [17] to turbulent fields.

In this paper, we extend the concept of explorability to turbulent fields using a framework described in [14] for source seeking in turbulent field. Since the measurements in the turbulent field are sporadic, the value of measurement collected by a robot is often zero or not acceptable. Thus, we compute expected value of rate of measurement using current information collected about the source and use it as an estimate of the rate of measurement at a location. Using the expected value of rate of measurement, we extent the notion of explorability to the turbulent scalar fields. We define the concept of coherent steps and establish the relationship between explorability of a field and coherent steps taken by a robot in the field. We propose a method to estimate the cumulative density function of the error in estimation of the rate of measurement and we use this technique to establish that the probability of making coherent steps is inversely proportional to the variance of the distribution of estimation error under the assumption that the estimation error is a Gaussian random variable. Finally we demonstrate using simulation techniques that the speed to converging to a source is directly proportional on the amount of information collected about the source which in turn potentially suggests that the probability of making coherent steps is inversely proportional to the variance in the error of estimation of rate of measurement.

The outline of the paper is as follows: we present the models used for motion, measurement and information in section II. The concept of explorability is presented in section III. In section IV we present the concept of coherent steps and establish the relationship between coherent steps and explorability of a turbulent field. A method of estimation of CDF of error in estimation of rate of measurement is

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presented in section V and using this estimation technique we find a relationship between probability of a coherent step and variance of the error in estimation. In section VI we present the simulation results to demonstrate the aforementioned relationship and finally in section VI we present the conclusion and future work.

II. Motion, sensing and information in a turbulent field

In this section we present the models used in [14] for a robot in a turbulent field. Suppose a robot is moving in a 2D space and collecting measurements of the field along its trajectory. We discretize the search space into uniform grids (see Fig. 1b). Let each grid point be denoted by h_i; $i = 1, 2, ..., N$ where N is the total number of grid points. Let the unknown source location be represented by a random variable $\boldsymbol{\theta} \in \mathbb{R}^2$. Let the robot's location be represented by the parameter $x \in \mathbb{R}^2$ and the measurement taken by the robot be represented by another random variable $\mathbf{Z} \in \mathbb{R}$.

The robot uses a motion model to move in the discretized space. At every location the robot stops for a fixed amount of time Δt to collect the measurement and update its information about the source. Then the robot uses the current information about the source to move to a next location using the control input generated by a source seeking algorithm.

A. Motion model

The motion dynamics of the robot is given by a particle model as follows:

$$x_{k+1} = x_k + u_k \tag{1}$$

Here x_k, u_k and x_{k+1} are the robot's current location, control input and robot's next location respectively.

B. Measurement model

1) Hits: In order to reduce the effect of the sporadic measurements in the study of turbulent fields, the concept of *hits* is introduced. A hit is defined as the detection of the field by the sensor. A hit is observed when the value of the field is above the detection threshold of the sensor (Fig. 1a).

2) Rate of hits: The rate of hits is the number of hits per unit time encountered by the robot at a location. Rate of hits is a function of the source and robot locations, hence it is a random variable. We assume that a deterministic function of rate of hits is available for the given type of the field. For example, Vergasolla et al. in [18], derived a deterministic function of rate of hits for the case of plume particles in turbulent field.

C. Information model

The random variable $\boldsymbol{\theta}$ has a probability distribution $p(\theta)$ which denotes the probability of source being at location $\boldsymbol{\theta} = \theta$. At the start of the trajectory of the robot, we initialize this distribution by a uniform distribution since we have no information about the source. At $t = k$, given the measurement z_k collected by the robot at x_k, we should be able to update $p(\theta)$. In order to be able to do that we will have to make the following assumptions.

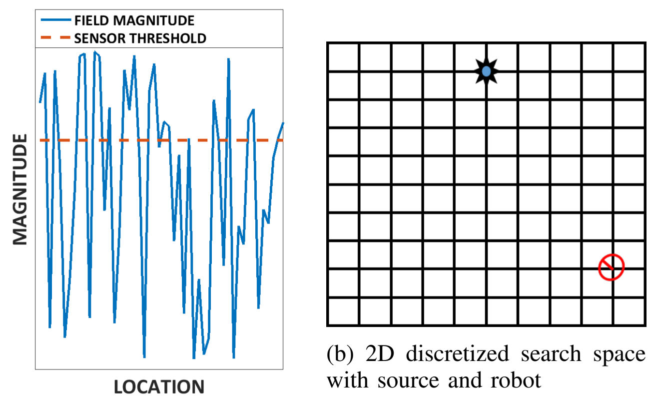
Assumption 2.1: Current measurement is dependent only on the current location of the robot and the source location and is independent of its trajectory and the corresponding measurements made along the trajectory.

$$p(z_k|\theta, z_{1:k-1}, x_{1:k}) = p(z_k|\theta, x_k) \tag{2}$$

Assumption 2.2: The probability distribution of the source location is independent of future robot location until the measurement is observed i.e.

$$p(\theta|z_{1:k-1}, x_{1:k}) = p(\theta|z_{1:k-1}, x_{1:k-1}) \tag{3}$$

Assumption 2.3: The number of hits encountered by a robot at a location is Poisson distributed and is independent of the number of hits encountered by it at any other location.



(b) 2D discretized search space with source and robot

(a) Field & sensor threshold

Fig. 1: Sensor reading in turbulent field and search space

Using the above assumptions, we present the different notions of probability distribution of source location as derived in [14].

1) Posterior distribution of source location: The posterior distribution of the source location $p(\theta|z_{1:k}, x_{1:k})$ is the current estimate of the source location given the current trajectory and the measurement collected along the trajectory. Posterior probability distribution can be computed as follows:

$$p_k^+(\theta) = p(\theta|z_{1:k}, x_{1:k}) \tag{4}$$
$$= \frac{p(\theta|z_{1:k-1}, x_{1:k-1}) \exp(-R(\theta, x_k)\Delta t) R(\theta, x_k)^{z_k}}{\int p(\theta|z_{1:k-1}, x_{1:k-1}) \exp(-R(\theta, x_k)\Delta t) R(\theta, x_k)^{z_k} d\theta}$$

2) A priori distribution of the source location: At $t = k$, the a priori distribution $p_{k+1}^-(\theta)$ is an estimate of the probability distribution of source location if the robot moves to $(\hat{x} = x_k + u_k)$ and might receive $\hat{z}_{k+1}$ measurement after waiting for Δt. Given, the posterior distribution $p_k^+(\theta)$, from (4), the a priori distribution can be computed as follows:

$$p_{k+1}^-(\theta) = p(\theta|\hat{z}_{k+1}, z_{1:k}, \hat{x}, x_{1:k}) \tag{5}$$
$$= \frac{p_k^+(\theta) \exp(-R(\theta, \hat{x})\Delta t) R(\theta, \hat{x})^{\hat{z}_{k+1}}}{\int p_k^+(\theta) \exp(-R(\theta, \hat{x})\Delta t) R(\theta, \hat{x})^{\hat{z}_{k+1}} dy}$$

3) Probability distribution of measurement: By assumption 2.3 we know that the number of hits are Poisson distributed. Now, given $\overline{R}_k^+(\hat{x}) = \int p_k^+(\theta)R(\theta,\hat{x})d\theta$ which is the expected value of rate of hits at location ($\hat{x} = x_k + u_k$) and the Poisson parameter $\Delta t\overline{R}_k^+(\hat{x})$. Probability distribution of measurement can be computed as follows:

$$p(\hat{z}_{k+1}|\hat{x}) = \frac{\exp(-\overline{R}_k^+(\hat{x})\Delta t)(\overline{R}_k^+(\hat{x}))^{\hat{z}_{k+1}}}{\hat{z}_{k+1}!} \tag{6}$$

III. Explorability of a turbulent scalar field

Since the measurements are available only in sporadic manner in a turbulent field therefore we use a probabilistic estimate of the rate of hits called predicted rate of hits as a measurement of the field .

A. Predicted rate of hits

Given the framework described in section II, the predicted rate of hits $\overline{R}_{k+1}^-(\hat{x})$ at a next possible robot location $\hat{x} = (x_k + u_k)$ can be defined as the expected value of $R(\theta,\hat{x})$ with respect to the probability distribution $p(\theta,\hat{z}_{k+1}|\hat{x},z_{1:k},x_{1:k})$.

$$\overline{R}_{k+1}^-(\hat{x}) = \int\int p(\theta,\hat{z}_{k+1}|\hat{x},z_{1:k},x_{1:k})R(\theta,\hat{x})dz\,d\theta \tag{7}$$

$$= \int\int p(\hat{z}_{k+1}|\hat{x},z_{1:k},x_{1:k})p_{k+1}^-(\theta)R(\theta,\hat{x})dz\,d\theta \tag{8}$$

Since the posterior distribution reflects the current information gathered by the robot about the source. It has certain divergence from the true distribution. This leads to an error in the estimate of the true rate of hits at a location.

Assumption 3.1: Let at time $t = k$ the posterior distribution of source location be given by $p_k^+(\theta)$ and the predicted rate of hits at location $\hat{x} = x_k + u_k$ be given as $\overline{R}_{k+1}^-(\hat{x})$. Let the true source location be given by θ^*, then the predicted rate of hits is an estimate of the true rate of hits such that

$$\overline{R}_{k+1}^-(\hat{x}) = R(\theta^*,\hat{x}) + \Lambda_{\hat{x}} \tag{9}$$

where $\Lambda_{\hat{x}}$ is a random variable which denotes estimation error.

The distribution of estimation error described in the above assumption depends on the divergence of the posterior from the true distribution. We will use this argument in section VI to show the relationship between variance in the error of estimation and probability of making coherent steps. Now in the next subsection we define explorability of a turbulent scalar field.

B. Explorability

Let at time $t = k$ the current location of the robot be $x_0 = x_k$ and $\mathbf{n}$ be an arbitrary vector of small norm. We can say that $x = x_0 + \mathbf{n}$ is a location in the neighborhood of x_0. Let $\epsilon > 0$ and $0 < \mathrm{p} < 1$ be two positive scalars and θ^* be the true location of the source. Let us define three sets around the location x_0:

$$U^+(\epsilon) = \{x|R(\theta^*,x) - R(\theta^*,x_0) \geq \epsilon\} \tag{10}$$

$$U^-(\epsilon) = \{x|R(\theta^*,x) - R(\theta^*,x_0) \leq -\epsilon\} \tag{11}$$

$$U^0(\epsilon) = \{x|\ |R(\theta^*,x) - R(\theta^*,x_0)| < \epsilon\} \tag{12}$$

A turbulent scalar field is said to be (p,ϵ) explorable at x_0 if one of the following conditions are satisfied:

If $x \in U^+(\epsilon)$, then

$$\Pr\left(\overline{R}_{k+1}^-(x) - \overline{R}_{k+1}^-(x_0) \geq \epsilon\right) > \frac{1+\mathrm{p}}{2} \tag{13}$$

If $x \in U^-(\epsilon)$, then

$$\Pr\left(\overline{R}_{k+1}^-(x) - \overline{R}_{k+1}^-(x_0) \leq -\epsilon\right) > \frac{1+\mathrm{p}}{2} \tag{14}$$

If $x \in U^0(\epsilon)$, then

$$\Pr\left(|\overline{R}_{k+1}^-(x) - \overline{R}_{k+1}^-(x_0)| < \epsilon\right) > \frac{1+\mathrm{p}}{2} \tag{15}$$

IV. Coherent steps in a turbulent field

In this section we extend the concept of coherent steps to turbulent scalar fields. We also show that if the field is locally explorable then the step made by the robot is coherent.

A. Coherent steps

The coherent steps of a robot in a turbulent field refers to a step made by the robot from location A to B when the estimate of the rate of hits (predicted rate of hits) at A and B are consistent with the actual rate of hits.

Definition 4.1: Let at time $t = k$, the current location of the search robot be $x_0 = x_k$ and the posterior probability distribution of source location be $p_k^+(\theta)$. Let Γ_{x_0} be a random variable given by $\Gamma_{x_0} = \Lambda_x - \Lambda_{x_0}$ and x be a location in the neighborhood of x_0 such that $x = x_0 + u_k$ for some u_k then the step of the robot to x is coherent if one of the following conditions are satisfied:

If $x \in U^+(\epsilon) \cup U^-(\epsilon)$, then

$$\Pr\left(|\Gamma_{x_0}| \leq \left|R(\theta^*,x) - R(\theta^*,x_0)\right| - \epsilon\right) > \frac{1+\mathrm{p}}{2} \tag{16}$$

If $x \in U^0(\epsilon)$, then

$$\Pr\left(|\Gamma_{x_0}| < \epsilon - \left|R(\theta^*,x) - R(\theta^*,x_0)\right|\right) > \frac{1+\mathrm{p}}{2} \tag{17}$$

B. Coherent steps in explorable turbulent field

For efficient source seeking we would like the robot to make as many coherent steps as possible. If the steps made by the robot are not coherent then the number of steps taken by the robot to localize the source will increase. As we have seen, the estimation of the rate of hits has error given by the value of Λ. Now we show that a robot in a explorable field will always generate coherent steps.

Theorem 4.1: Let at time $t = k$, the current location of the search robot be $x_0 = x_k$ and the posterior probability distribution of source location be $p_k^+(\theta)$. Let $x = x_0 + u_k$ be a step of the robot such that x is in the neighborhood of x_0. Now, if the turbulent scalar field is (p,ϵ) explorable at x_0 then the step taken by the robot is coherent.

Proof: Since x can be in any of the three sets defined in (13), (14) and (15) therefore we will take each case one by one and prove the theorem for each of them.

Case 1: $x \in U^+(\epsilon)$

Since, the turbulent scalar field is (p,ϵ) explorable at x_0

therefore using the definition of explorability of a turbulent scalar field in (13) we have:

$$\Pr\left(\overline{R}^{-}_{k+1}(x) - \overline{R}^{-}_{k+1}(x_0) \geq \epsilon\right) > \frac{1+\mathrm{p}}{2}$$

Using (9) we can say that

$$\Pr\left(R(\theta^*, x) + \Lambda_x - R(\theta^*, x_0) - \Lambda_{x_0} \geq \epsilon\right) > \frac{1+\mathrm{p}}{2}$$

We know that $\Gamma_{x_0} = \Lambda_x - \Lambda_{x_0}$, therefore we have:

$$\Pr\left(R(\theta^*, x) - R(\theta^*, x_0) + \Gamma_{x_0} \geq \epsilon)\right) > \frac{1+\mathrm{p}}{2}$$

Rearranging the terms we get

$$\Pr\left(\Gamma_{x_0} \geq -\left(\left(R(\theta^*, x) - R(\theta^*, x_0)\right) - \epsilon\right)\right) > \frac{1+\mathrm{p}}{2}$$

Using, (13), we know that $R(\theta^*, x) - R(\theta^*, x_0) > 0$ thus, the above equation satisfies the following equation for $\Gamma_{x_0} < 0$:

$$\Pr\left(|\Gamma_{x_0}| \leq \left|R(\theta^*, x) - R(\theta^*, x_0)\right| - \epsilon\right) > \frac{1+\mathrm{p}}{2}$$

Using (16), we can say that the step taken by the robot is coherent.

Case 2: $x \in U^{-}(\epsilon)$

Again, using (13) and (9) and the definition of Γ_{x_0} we have:

$$\Pr\left(R(\theta^*, x) - R(\theta^*, x_0) + \Gamma_{x_0} \leq -\epsilon)\right) > \frac{1+\mathrm{p}}{2}$$

Rearranging the terms we get

$$\Pr\left(\Gamma_{x_0} \leq R(\theta^*, x_0) - R(\theta^*, x) - \epsilon\right) > \frac{1+\mathrm{p}}{2}$$

Using, (14), we know that $R(\theta^*, x_0) - R(\theta^*, x) > 0$ thus, the above equation satisfies the following equation for $\Gamma_{x_0} > 0$:

$$\Pr\left(|\Gamma_{x_0}| \leq \left|R(\theta^*, x) - R(\theta^*, x_0)\right| - \epsilon\right) > \frac{1+\mathrm{p}}{2}$$

Using (16), we can say that the step taken by the robot is coherent.

Case 3: $x \in U^{0}(\epsilon)$

Using (13) we have

$$\Pr\left(\left|\overline{R}^{-}_{k+1}(x) - \overline{R}^{-}_{k+1}(x_0)\right| < \epsilon\right) > \frac{1+\mathrm{p}}{2}$$

First we let $\overline{R}^{-}_{k+1}(x) - \overline{R}^{-}_{k+1}(x_0) > 0$, then we have

$$\Pr\left(\overline{R}^{-}_{k+1}(x) - \overline{R}^{-}_{k+1}(x_0) < \epsilon\right) > \frac{1+\mathrm{p}}{2}$$

Now using (9) and the definition of Γ_{x_0} we have:

$$\Pr\left(\Gamma_{x_0} < \epsilon - \left(R(y^*, x) - R(y^*, x_0)\right)\right) > \frac{1+\mathrm{p}}{2} \qquad (18)$$

Now we let $\overline{R}^{-}_{k+1}(x) - \overline{R}^{-}_{k+1}(x_0) < 0$, then we have

$$\Pr\left(-\left(\overline{R}^{-}_{k+1}(x) - \overline{R}^{-}_{k+1}(x_0)\right) < \epsilon\right) > \frac{1+\mathrm{p}}{2}$$

Now using (9) and the definition of Γ_{x_0} we have:

$$\Pr\left(\Gamma_{x_0} > -\left(\epsilon - \left(R(y^*, x_0) - R(y^*, x)\right)\right)\right) > \frac{1+\mathrm{p}}{2} \qquad (19)$$

Now, comparing the results of (18) and (19) with (17) we can say that the step taken by the robot is coherent. ■

V. Analysis of Error Distribution

In this section we try to compute the probability of the error Γ_{x_0} being bounded by computing its cumulative density function (CDF). Let at time $t = k$, the robot be at $x_0 = x_k$ and the posterior distribution be $p^{+}_{k}(\theta) = p(\theta|z_{1:k}, x_{1:k})$. Let x be one of the possible next steps of the robot. Then using Assumption 3.1, for any location x in the search space we can say that:

$$R^{-}_{k+1}(x) = R(\theta^*, x) + \Lambda_x \qquad (20)$$

Let us consider that the true location of the source is θ^* and the current location of the robot is x_0 then the true rate of hits at the current location of the robot can be given by $R(\theta^*, x_0)$. The true distribution of the source location can be given as a delta distribution $T(\theta)$ which can be approximated using a Gaussian distribution with small variance.

Assumption 5.1: The true distribution of source location can be given as:

$$T(\theta) \sim \mathcal{N}(0, \varepsilon^2)\ ;\ \varepsilon^2 \ll 1 \qquad (21)$$

Now using (8), (20) and Assumption 5.1, we have:

$$\iint p(\theta, \hat{z}_{k+1}|x, z_{1:k}, x_{1:k}) R(\theta, x) dz d\theta = \int T(\theta) R(\theta, x) d\theta + \Lambda_x \qquad (22)$$

After rearranging the integrals we have:

$$\Lambda_x = \int \left(\int p(\theta, \hat{z}_{k+1}|x, z_{1:k}, x_{1:k}) dz - T(\theta)\right) R(\theta, x) d\theta \qquad (23)$$

Let for simplicity the condition $\{z_{1:k}, x_{1:k}\}$ be given by the condition $\{\mathcal{A}\}$. Now, the distribution $p(\theta, \hat{z}_{k+1}|x, \mathcal{A}) = p(\theta, \hat{z}_{k+1}|x, z_{1:k}, x_{1:k})$ can be obtained as follows:

$$\begin{aligned} p(\theta, \hat{z}_{k+1}|x, \mathcal{A}) &= p(\theta|\hat{z}_{k+1}, x, \mathcal{A}) p(\hat{z}_{k+1}|x, \mathcal{A}) \\ &= \frac{p(\hat{z}_{k+1}|\theta, x, \mathcal{A}) p(\theta|x, \mathcal{A})}{p(\hat{z}_{k+1}|x, \mathcal{A})} p(\hat{z}_{k+1}|x, \mathcal{A}) \\ &= p(\hat{z}_{k+1}|\theta, x, \mathcal{A}) p(\theta|x, \mathcal{A}) \end{aligned} \qquad (24)$$

Using Assumption 2.2, we can say that $p(\theta|x, \mathcal{A}) = p^{+}_{k}(\theta)$, thus the above equation becomes:

$$p(\theta, \hat{z}_{k+1}|x, \mathcal{A}) = (\hat{z}_{k+1}|\theta, x, \mathcal{A}) p^{+}_{k}(\theta) \qquad (25)$$

Using (23) and (25), we have

$$\begin{aligned} \Lambda_x &= \int \left(\int p(\hat{z}_{k+1}|\theta, x, \mathcal{A}) p^{+}_{k}(\theta) dz - T(\theta)\right) R(\theta, x) d\theta \\ &= \int \left(\int p(\hat{z}_{k+1}|\theta, x, \mathcal{A}) dz - \frac{T(\theta)}{p^{+}_{k}(\theta)}\right) p^{+}_{k}(\theta) R(\theta, x) d\theta \end{aligned} \qquad (26)$$

Using Assumption 2.3, we can say that $\int p(\hat{z}_{k+1}|\theta, x, \mathcal{A})dz = \Delta t R(\theta, x)$. Using this we have

$$\Lambda_x = \int \left(\Delta t R(\theta, x) - \frac{T(\theta)}{p_k^+(\theta)} \right) p_k^+(\theta) R(\theta, x) d\theta$$

$$\Lambda_x = \Delta t \int \left(R^2(\theta, x) p_k^+(\theta) - \frac{1}{\Delta t} R(\theta, x) T(\theta) \right) d\theta \quad (27)$$

Using the above equation, we can say that:

$$\Lambda_x = \Delta t \left(\mathbb{E}_{p_k^+(\theta)} \left[R^2(\theta, x) \right] - \frac{1}{\Delta t} \mathbb{E}_{T(\theta)} \left[R(\theta, x) \right] \right) \quad (28)$$

We know that $\Gamma_{x_0} = \Lambda_x - \Lambda_{x_0}$, using (28) we have:

$$\Gamma_{x_0} = \Delta t \Bigg(\mathbb{E}_{p_k^+(\theta)} \left[R^2(\theta, x) - R^2(\theta, x_0) \right] - \frac{1}{\Delta t} \mathbb{E}_{T(\theta)} \left[R(\theta, x) - R(\theta, x_0) \right] \Bigg) \quad (29)$$

We introduce the function $\pi(\epsilon)$:

$$\pi(\epsilon) = \begin{cases} \left| R(\theta^*, x) - R(\theta^*, x_0) \right| - \epsilon \; ; x \in U^+(\epsilon) \cup U^-(\epsilon) \\ \epsilon - \left| R(\theta^*, x) - R(\theta^*, x_0) \right| \; ; x \in U^0(\epsilon) \end{cases} \quad (30)$$

Now, using (29) we have:

$$\Pr \left(|\Gamma_{x_0}| \leq \pi(\epsilon) \right) = \Pr \Bigg(\Bigg| \mathbb{E}_{p_k^+(\theta)} \left[R^2(\theta, x) - R^2(\theta, x_0) \right] - \frac{1}{\Delta t} \mathbb{E}_{T(\theta)} \left[R(\theta, x) - R(\theta, x_0) \right] \Bigg| \leq \frac{\pi(\epsilon)}{\Delta t} \Bigg) \quad (31)$$

We can see that finding the CDF of the estimation error is a challenging problem if the only knowledge we have is the trajectory and the corresponding measurement obtained by the robot. In order to simplify the problem we make further assumptions about the noise in estimation of rate of hits.

Assumption 5.2: Let the error in estimation of the rate of hits at a location x' through predicted rate of hits $R_{k+1}^-(x')$ be a zero mean Gaussian random variable with variance $\sigma_{x'}^2$.

$$R_{k+1}^-(x') = R(\theta^*, x') + \Lambda_{x'} \quad ; \Lambda_{x'} \sim \mathcal{N}(0, \sigma_{x'}^2) \quad (32)$$

Theorem 5.1: Let at time $t = k$, a robot be at location $x_k = x_0$ such that that posterior distribution of source location be $p_k^+(\theta)$. Let $x = x_0 + u_k$ be a step of the robot such that x is in the neighborhood of x_0. Let the assumptions 3.1 and 5.2 hold such that variance at x_0 and x be $\sigma_{x_0}^2$ and σ_x^2 respectively. Now, the probability of step taken by the robot being coherent is inveresely proportional to $\sqrt{\sigma_x^2 + \sigma_{x_0}^2}$.

Proof: Since Λ_{x_0} and Λ_x are zero mean Gaussian random variables with variance $\sigma_{x_0}^2$ and σ_x^2 respectively therefore $\Gamma_{x_0} = \Lambda_x - \Lambda_{x_0}$ is also a normally distributed random variable with zero mean and variance $\sqrt{\sigma_x^2 + \sigma_{x_0}^2}$. Now, the cumulative probability distribution of random variable Γ_{x_0} can be given as:

$$\Pr \left(\Gamma_{x_0} \leq \pi(\epsilon) \right) = \frac{1}{2} \left[1 + \text{erf} \left(\frac{\pi(\epsilon)}{\sqrt{2(\sigma_x^2 + \sigma_{x_0}^2)}} \right) \right] \quad (33)$$

Using the above CDF, we have:

$$\Pr \left(|\Gamma_{x_0}| \leq \pi(\epsilon) \right) = \text{erf} \left(\frac{\pi(\epsilon)}{\sqrt{2(\sigma_x^2 + \sigma_{x_0}^2)}} \right) \quad (34)$$

Now using (16), (17) and (34) we can see that probability of x being a coherent step is inveresely proportional to $\sqrt{\sigma_x^2 + \sigma_{x_0}^2}$. ■

Remark 5.1: Since explorability guarantees coherent steps, using (16), (17) and (34) we can say that the explorable probability p is inversely proportional to the variance of estimation error. If p is large then the variance of the estimation error is small and vice versa.

From (34), it is apparent that the variance of the random variable Λ is dependent on the current information about the source location. As mentioned before, the amount of current information on the source can be realized if we know the divergence between the true distribution and the posterior distribution of the source location. Since it is not possible to compute the divergence (because the source location is unknown), we resort to numerical simulation to show that if we gather more information about the source then we are more probable to make coherent steps.

VI. Simulation results

As stated before, the variance of the random variable Λ around value zero is inversely proportional to the information gained about the source by the robot. This in turn means the lesser the divergence between the posterior and the true field the lesser the variance of random variable Λ.

Remark 6.1: If a robot collects more measurement in the space at various locations then the variance of Λ around zero value decreases and hence using Theorem 5.1 the probability of making coherent steps increases.

We show the validity of Remark 6.1 using simulations results. We deploy a group of robots in the search space to localize a source using expected rate algorithm [14]. Each robot, at every step, shares the measurement and location with all other robots if it receives a non-zero measurement. In this way each robot has more information about the source compared to the case if it was trying to localize the source alone. If we show that the group can localize the source in less number of steps than any single agent then we can demonstrate the validity of Remark 6.1.

We simulate a 2D search space with a plume source which generates a turbulent field and we discretize the search space into grids of equal sizes. The source is represented by a red asterisk (on top). The plume particles emitted by the source have a diffusivity of 1 and a mean life of 2500 seconds. The wind is blowing with a speed of $1 m/s$ in the negative y-axis direction. The robot stops at every grid point in its trajectory for 5 seconds to collect the measurement. We deploy three robots in a search space to localize a source of plume generating turbulent plume field. The robots first search in a group by sharing their measurement and location (if measurement greater than zero). The starting point of each robot is represented by a black asterisk (at bottom) and the path of the robots is shown in Figure 2. In the second

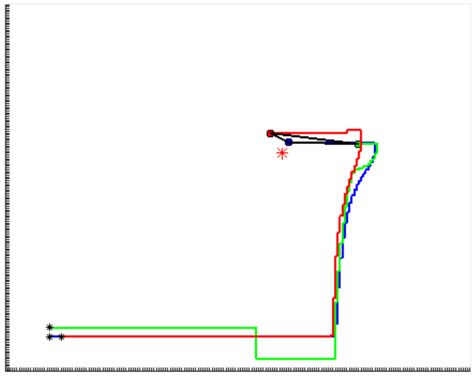
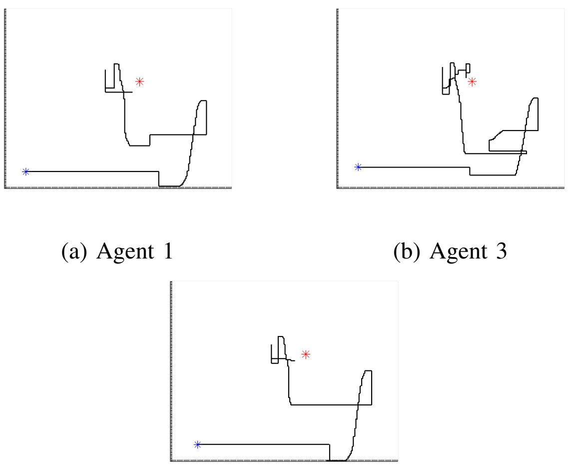


Fig. 2: Multiple agents localizing the source in a team



(a) Agent 1

(b) Agent 3

(c) Agent 3

Fig. 3: Single agents localizing the source

round the robots were deployed individually with exactly same initial positions. The starting point of each robot is represented by a blue asterisk (at bottom) and the path of each robot is shown in Figure 3. The number of steps taken by Agent 1, 2 and 3 were 570, 731 and 597 respectively. However, the number of steps taken by the agents when working in a team was 284. We can clearly observe that the number of steps taken by agents when working in a team is significantly less than individual robots working alone. This indicates that the estimate of the rate of hits (cost function for source seeking) in case of agents working in a team was better than the cases when the robots worked individually. Hence, it reinstates our claim that the probability of coherent steps is inversely proportional to the variance of the error in estimation of rate of hits.

VII. Conclusion and Future work

In this paper, we extended the notion of explorabilty of noisy scalar fields to turbulent scalar fields. We define the coherent steps of a robot in a turbulent field and derived the bounds on the error to ensure coherent steps with high probability. We also proved that the variance of error in estimation around zero is inversely proportional to the probability of making coherent steps and demonstrated the same using simulation results. In future we would like to prove the convergence of source seeking algorithms in the turbulent field using the notion of explorabiltiy.

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